Remarks on a flame propagation model

Guy Barles

INRIA

CENTRE
SOPHIA ANTIPOLIS

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105
78153 Le Chesnay Cedex
France
Tél.:(3) 954 90 20

Rapports de Recherche

N°451

REMARKS ON A FLAME PROPAGATION MODEL

Guy BARLES

Novembre 1985

Remarks on a flame propagation model
(Remarques sur un modèle de propagation de flamme)

Abstract : Two years ago, J. Sethian studied the propagation of a flame front which was assumed to move in a direction normal to itself with constant speed. The aim of this work is to compare both the problems and the methods, used by J. Setthian to solve them for this model of flame propagation with the ones which appear in Geometrical Optics, Optimal Control and first order Hamilton Jacobi Equations.

Key words : flame front propagation, ignition curves, characteristics, geometrical optics, optimal control, first order Hamilton Jacobi Equations.

Résumé : Il y a deux ans, J. Sethian a étudié la propagation d'un front de flamme dont la vitesse de chaque point était supposée constante et dirigée suivant la normale au front en ce point. Le but de ce travail est de comparer à la fois les problèmes et les méthodes, utilisées par J. Sethian pour les résoudre, avec ceux qui interviennent en Optique Géométrique, en Contrôle Optimal et dans les équations de Hamilton Jacobi du premier ordre.

Mots-clé : propagation du front de flamme, courbes d'ignition, caractéristiques, optique géométrique, contrôle optimal, équation de Hamilton Jacobi du premier ordre.

REMARKS ON A FLAME PROPAGATION MODEL

Guy BARLES

Introduction.

In [24], J. Sethian introduces a model of flame propagation and, to investigate the shape of the flame front, he defines the notion of ignition curves and an entropy condition. Our aim is to make some remarks on this model and on the methods used by J. Sethian to study it in the setting of geometrical optics, of optimal control and of the theory of first order Hamilton Jacobi Equations, and particularly using the notion of viscosity solution introduced by M.G. Crandall and P.L. Lions in [8].

We first briefly recall the model of flame propagation : as in [24], we consider Γ a simple, closed, regular parametrized curve in $\mathbb{R}^2$. The region outside Γ is filled with a premixed combustible fluid and the particles inside are burnt. At time $t = 0$, the particles along Γ are ignited : then we want to know the position of the flame front at all time $t > 0$. As in [24], we assume that the flame front propagates in a direction normal to itself with constant speed c.

Our aim is to compare both the problems and the methods, used to solve them in [24], for this model of flame propagation with the ones which appear in geometrical optics, optimal control and first order Hamilton Jacobi equations. Let us just mention that geometrical optics and optimal control are closely related with first order Hamilton Jacobi equations and provide to their studies a lot of methods as we shall show.

In section I, we present in details the model and we state the problems to be solved in order to determine the position of the flame front at each time $t > 0$. Then we briefly recall the notion of ignition curves and the entropy condition introduced by J. Sethian . Finally, we give a new "weak" formulation of the problem by using the indicator function φ of the burnt region. We call this formulation "weak" since we no more need the existence of the normal vector at each point of Γ.

The section II is devoted to show the connections between the model of flame propagation and geometrical optics. More precisely, if we take a convenient Lipschitz regularization of the indicator function of the burnt region, which we still denote by φ and which you can imagine to represent the mass function of burnt gas, we obtain that the propagation of the curves $\{\varphi = c^t\}$ is analogous to the propagation of waves surfaces in some model of geometrical optics. Therefore, the mass fraction of burnt gas satisfies an equation of Eikonal type

$$\frac{\partial\varphi}{\partial t} - c \cdot |D\varphi| = 0 \tag{1}$$

In section III, we briefly introduce the notion of viscosity solution for first order Hamilton Jacobi Equations introduced by M.G. Crandall and P.L. Lions in [8]. Then, we show some relations between first order Hamilton Jacobi Equations and Geometrical Optics which permit us to prove that φ is a viscosity solution of (1).

In section IV, we give another proof of the fact that φ is a viscosity solution of (1) by using some Optimal Control methods, in particular the Dynamic Programming Principle (cf. [10], [19]) . It is worth noting that Optimal Control is closely related with first order Hamilton Jacobi Equations.

In section V, we consider the simpler case when Γ is given by $\{(y,x) \,/\, y = u_o(x)\}$, u_o at least continuous. We prove that, at each time $t > 0$, the flame front can be parametrized by $y = u(x,t)$ where u is the unique viscosity solution of

(2) $$\frac{\partial u}{\partial t} - c \cdot \sqrt{1+|\nabla u|^2} = 0 \; .$$

We also study the directions of propagation when u_o is not smooth enough. For that, we build "generalized ignition curves" ; roughly speaking, this construction is similar to the one given by J. Sethian in [24] to study the evolution of convex, non smooth flame fronts but is a little more general.

Finally in section VI, we make some remarks, using the viscosity solution formulation, in particular for the stability of the flame front, its asymptotic behaviour, some possible generalizations (for example, to non-constant propagation speed) and for numerical simulation. Let us finally mention that this formulation allows us to investigate flame fronts propagation in $\mathbb{R}^3$ with no more difficulties than in $\mathbb{R}^2$.

Our plan is as follows

I. Setting of the problem.

We want to study the notion of a flame front, propagating in a premixed, combustible fluid with no boundaries under the same physical assumptions as in [24] . Thus, we consider the following simplified model.

Let Γ be a simple, closed, "regular", parametrized curve in $\mathbb{R}^2$; we assume that the region outside Γ is filled with a premixed, combustible fluid and that the particles inside Γ are burnt. At time $t = 0$, we ignite the particles along Γ ; then the boundary between burnt and unburnt particles changes. As in [24], we suppose that the flame front propagates in a direction normal to itself with constant speed c . So, if we denote by $(X(s,t)\ ,\ Y(s,t))$ the position of the front at time t , X and Y must satisfy

$$(3)\qquad \begin{cases} X_t = \dfrac{-\,c\cdot Y_s}{(X_s^2 + Y_s^2)^{1/2}} \\[2ex] Y_t = \dfrac{c\cdot X_s}{(X_s^2 + Y_s^2)^{1/2}} \end{cases}$$

Now if $\Gamma = \{(X(s,0)\ ,\ Y(s,0))\}$ is sufficiently smooth, we know by [24] that, for t small enough, the position of the flame front is given by

$$(4)\qquad \begin{cases} X(s,t) = X(s,0) - \dfrac{c\cdot t\cdot Y_s(s,0)}{[\,(X_s(s,0))^2+(Y_s(s,0))^2]^{1/2}} \\[2ex] Y(s,t) = Y(s,0) + \dfrac{c\cdot t\cdot X_s(s,0)}{[\,(X_s(s,0))^2+(Y_s(s,0))^2]^{1/2}} \end{cases}$$

The straight lines $t \to (X(s,t),(Y(s,t))$ are called the <u>ignition curves</u>.

As we have seen above, they enable us to determine the position of the flame front for t small enough if Γ is smooth. But if t is not small enough, ignition curves can collide and we must indicate how the motion of the flame front continues beyond the point of the first collision. Another question, closely related to the preceding one, is to know what happens when the flame front is not smooth enough (i.e. when the normal vector is not well-defined). As J. Sethian remarks, this problems are similar to those which appear in the Characteristic method for hyperbolic systems (cf. [6], [14], [15], [18], for example) : the fact that the solution does not remain smooth (in general) even when the initial condition is smooth implies that we can not defined globally the solution by the Characteristic method.

In [24], J. Sethian answers these questions by introducing an entropy criterion :

A propagation flame front satisfies the entropy criterion if once a particle burns, it remains burnt.

In fact, he gives a "weak" formulation of the propagation by introducing the indicator functions of the burnt region i.e.

$$\varphi(x,y,t) = \begin{cases} 1 & \text{if} \quad (x,y) \text{ is burnt at time } t = 0 \\ 1 & \text{if} \quad (ct)^2 \geqslant \operatorname*{Min}_s \left[(x-X(s,0))^2+(y-Y(s,0))^2 \right] \\ 0 & \text{otherwise} \end{cases}$$

This formulation is said to be "weak" because it requires less regularity of Γ than the preceding formulation but it obviously gives the same flame front for small $t > 0$ when Γ is smooth .

Now if we denote by $\overline{\Omega}$ the burnt region at time $t = 0$ (the region inside Γ), we have :

(5) $$\varphi(x,y,0) = 1_{\overline{\Omega}}(x,y)$$

and

(6) $$\varphi(x,y,t) = \sup_{|x-x_1|^2+|y-y_1|^2 \leqslant (ct)^2} (\varphi(x_1,y_1,0))$$

Since we are only interested in the shape of the burnt region $\{(x,y) \ / \ \varphi(x,y,t) = 1\}$, we are going to replace (5) by

(7) $$\varphi(x,y,0) = (1 - d(x,y) \ , \ \overline{\Omega}))^+ + d((x,y) \ , \ \Omega^c)$$

to have more regularity for $\varphi(x,y,0)$ especially on Γ .

From now on, we shall denote by x the couple (x,y) .

Remark I.1 : Notice that this "weak" formulation allows us to consider more general flame front propagation : indeed we can take every closed set Ω , which may be not connected or not bounded, as the initial burnt region. Moreover, we shall see that the study is not more difficult in $\mathbb{R}^3$ than in $\mathbb{R}^2$.

Remark I.2 : From (6) and (7), we see that the burnt region at time t is $\{(x,y) \ / \ \varphi(x,y,t) \geqslant 1\}$ and the flame front is $\{(x,y) \ / \ \varphi(x,y,t) = 1\}$.

II. Flame front propagation and geometrical optics :

We are going to show the relations between the model of flame propagation described above and geometrical optics by a simple example. For more details on geometrical optics, we refer to [5] and [17]. Let us just mention that geometrical optics methods occur in the study of Hamilton Jacobi equations and particularly for viscosity solutions as we shall see in the next section.

We consider the following problem :

$$(8)\qquad \begin{cases} \dfrac{\partial^2 u^\varepsilon}{\partial t^2} - c^2 \cdot \Delta u^\varepsilon = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \qquad (N = 2 \text{ or } 3) \\ u^\varepsilon(x,0) = \varepsilon e^{iS(x)/\varepsilon} \cdot v_o(x) & \text{in } \mathbb{R}^N \\ \dfrac{\partial u^\varepsilon}{\partial t}(x,0) \quad \text{given} & \end{cases}$$

u^ε is a complex valued function which represents either the electric field or the magnetic field, S and v_o are real valued functions, $\varepsilon > 0$ is small.

The aim of geometrical optics is to study the propagation of the electric or magnetic field in the case of short waves length i.e. when $\varepsilon \to 0$. Largely inspired by [5], we are going to construct an expansion of u^ε of the form

$$u^\varepsilon(x,t) = \varepsilon e^{iS(x,t)/\varepsilon}\, v^\varepsilon(x,t)$$

where $v^\varepsilon(x,t) = v_o(x,t) + \varepsilon v_1(x,t) + \ldots.$

But we restrict ourselves to the first term of this expansion in ε, i.e. :

$$u^{\varepsilon}(x,t) = \varepsilon e^{iS(x,t)/\varepsilon} \cdot v_o(x,t) \tag{9}$$

Inserting (9) in (8) yields :

$$(S_t)^2 - c^2(\nabla S)^2 = 0 \qquad \text{(terms of order } O(\varepsilon^{-1})\text{)} \tag{10}$$

$$S_{tt}v_o + 2S_t v_{ot} - c^2 \Delta S v_o - 2c^2 \nabla \cdot S \cdot \nabla v_o = 0 \qquad \text{(terms of order } O(1)\text{)} \tag{11}$$

(10) is called the <u>Eikonal equation,</u> (11) is the <u>Transport equation</u>.

Now let us consider the solution S satisfying

$$S_t - c|\nabla S| = 0 \tag{12}$$

By analogy with the plane wave case where $S(x,t) = w \cdot t - k \cdot x$, we define w and k by

$$\begin{cases} w = S_t(x,t) \\ k = -\nabla S(x,t) \end{cases}$$

So that (12) yields

$$w(k) = c|k|$$

which is the usual law of propagation in the vaccuum (cf. [17]). To study (12), it is physically natural (cf. [17]) to introduce the <u>Hamiltonian's system</u> of ODE for <u>the rays $x(t)$</u> and <u>the wave numbers $k(t)$</u>

$$\begin{cases} \dfrac{dx}{dt}(t) = \dfrac{\partial w}{\partial k}(k(t),x(t)) \quad , \quad x(0) = x_o \\[2ex] \dfrac{dk(t)}{dt} = -\dfrac{\partial w}{\partial x}(k(t),x(t)) \quad , \quad k(0) = k_o \end{cases} \tag{13}$$

Since w does not depend on x , we deduce :

$$(14)\qquad \begin{cases} k(t) = k_o \\ x(t) = x_o + c\,\dfrac{k_o}{|k_o|}\,t \end{cases}$$

Now imagine that $S(x)$ is equal to $\varphi(x,0)$ defined in part I . Since $k_o = -\nabla\varphi(x_o,0)$, if Γ is smooth and if $x_o \in \Gamma$, since $\varphi(x,0)$ is constant on Γ , k_o belongs to the normal at x_o and is oriented outside $\overline{\Omega}$; then $\dfrac{k_o}{|k_o|}$ is the unit outward normal and (14) represents exactly ignition curves. So the propagation of Γ is the same in the model of flame propagation and in the one of geometrical optics ; the curve moves normal to itself at constant speed. In geometrical optics, the curves $\{S = c^t\}$ are called the waves surfaces. So the waves surfaces move normal to themselves with constant speed c .

Let us just remark that (13) is not well-defined if $\nabla S(x_o) = 0$. As long as S is smooth, we have

$$\begin{aligned} \frac{d}{dt}\,(S(x(t),t)) &= \frac{\partial S}{\partial t}(x(t),t) + \dot{x}\cdot\nabla S(x(t),t) \\ &= w - \frac{\partial w}{\partial k}\cdot k \\ &= 0 \end{aligned}$$

Hence as long as S is smooth, we have

$$(15)\qquad S(x(t),t) = S(x(0))$$

<u>Remark II.1</u> : In geometrical optics, one usually uses the analogy with Mecanics (cf. [16] and [17]). In mecanics, the Action S is solution of a first-order Hamilton Jacobi equation

$$\frac{\partial S}{\partial t} + H(x,\nabla S,t) = 0$$

where H is the Hamiltonian function of the particle, $p = \nabla S$ its impulse and S is associated to the Lagrangian $p \cdot \frac{\partial H}{\partial p} - H$. Therefore, using the analogy, we have, in particular, that the Lagrangian in geometrical optics is given by $k \cdot \frac{\partial w}{\partial k} - \omega$ which is identically 0 . This is a consequence that the propagation of rays is analogous to the motion of zero mass particles.

But in the same way as ignition curves collide, it may happen that the mapping $x_t : x_o \to x(t)$ does not remain a C^1 diffeomorphism, and in particular does not remain injective, such that S is no more defined by (15). This implies in particular that S does not remain smooth.

Remark II.2 : We now explain this phenomena in a physical way (cf. [5] and [7]). Multiplying (11) by $\bar{v}_o$ and adding to the conjugate of (11) multiplied by v_o , we obtain

$$(16) \qquad (w|v_o|^2)_t - \nabla\cdot(c^2\cdot k\cdot|v_o|^2) = 0$$

Since $w_t = \frac{\partial w}{\partial k} \cdot k_t$ and $k = -\nabla S$, we have

$$w_t = -\frac{\partial w}{\partial k} \cdot \nabla w$$

Hence :

$$(17) \qquad (w^2|v_o|^2)_t - \nabla\cdot\left(w^2\cdot\frac{\partial w}{\partial k}\cdot|v_o|^2\right) = 0$$

$w^2|v_o|^2$ is the wave energy. From (17), we get the usual law of energy propagation of geometrical optics (see [5] for details) : the energy density is inversely proportional to the density of rays at the point : i.e.

$$(18) \qquad (w^2|v_o|^2)\ (x(t),t) = (w^2|v_o|^2)\ (x_o,0) \cdot [J(x_t)]^{-1}$$

where $J(x_t)$ is the Jacobian determinant of the transformation x_t .

Since w is constant and non zero along the trajectory $x(t)$, it follows that energy becomes singular at the points where $J(x_t) = 0$: these points are called caustic points. Physically, at these points, the energy is very important but remains finite ; in fact, the approximation of geometrical optics is inadequate. Mathematically, expansion (9) fails.

We come back to the study of (12). In geombtrical optics, it is natural to introduce the Lagrangian function defined by

$$(19) \qquad L(\dot{x}) = \sup_k ((\dot{x}|k) - w(k))$$

which is the dual convex function of w (cf. [5]).

Ramark II.1 and the computation of $\frac{d}{dt}(S(x(t),t))$ above shows that the "Action" associated to L is $-S$. Then, geometrical optics laws give

$$(20) \qquad (-S)(x,t) = \inf_{\{\xi \mid \xi(t) = x\}} \left[(-S)(\xi(0)) + \int_0^t L(\dot{\xi}(s))ds\right]$$

the infimum being taken over all differentiable paths ξ which terminate at x at time t . S given by (20) is the unique "smooth" solution of (12) up to the time when caustics form. Observe that, in our case :

$$L(\dot{x}) = \begin{cases} 0 & \text{if } |\dot{x}| \leqslant c \\ +\infty & \text{otherwise} \end{cases}$$

Hence

$$(-S)(x,t) = \inf_{\substack{\xi(t) = x \\ |\dot{\xi}| \leqslant c. \ \text{pp}}} (-S(\xi(0)))$$

Since $\xi(0)$ can be any of the points y such that $|y-x| \leqslant ct$, we get :

(21) $$S(x,t) = \sup_{|y-x| \leqslant ct} (S(y))$$

which is exactly (6).

In the next section, we briefly present the notion of viscosity solutions for first order Hamilton-Jacobi Equations but let us already point out that S given by (21) is the unique viscosity solution of (12).

III. Some elements on the notion of viscosity solutions for first-order Hamilton Jacobi Equations.

We give a brief introduction of the notion of viscosity solutions for first order Hamilton Jacobi Equations in the case of initial value problems and a summary of its properties for the sake of completeness and of clarity. A complete presentation can be found in [8] , [7] or [19] .

We are interested in the following problems

$$(22)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + H(x,t,u,Du) = 0 & \text{in} \quad \mathbb{R}^N \times (0,\infty) \\ u(x,0) = u_o(x) & \text{in } \mathbb{R}^N \end{cases}$$

which is called the Cauchy problem for Hamilton Jacobi Equations. Let us first mention that these types of problems occur in Mecanics, Optics and Optimal control.

It is well-hnown that, in general, (22) fails to have classical solutions - i.e. of class C^1 - even when u_o is very smooth. On the other hand, several authors proved the existence of generalized solutions - i.e. in $W^{1,\infty}_{loc}(\mathbb{R}^N \times (0,T))$ - which satisfy the equations almost everywhere. But in general (22) has many generalized solutions and this lack of uniqueness is a problem in view of applications. Moreover, to prove existence results, one usually applies the vanishing viscosity method i.e. approximating (22) by

$$(23)\qquad \frac{\partial u^\varepsilon}{\partial t} - \varepsilon\cdot\Delta u^\varepsilon + H(x,t,u^\varepsilon,Du^\varepsilon) = 0 \ .$$

However, due to the presence of the non-linear term H , the analysis of the limit when $\varepsilon \to 0$ requires the knowledge of numerous and sharp a priori estimates which are very difficult to derive, and even not necessarily valid in the general case.

Nevertheless it is obvious that the "good" solution of (22) (if it exists) must be the limit (in some sense) of u^ε .

The idea of M.G. Crandall and P.L. Lions is to use the Maximum Principle : let u^ε a smooth solution of (23) and let $\varphi \in C^\infty(\mathbb{R}^N \times (0,\infty))$; if $(x_\varepsilon, t_\varepsilon)$ is a local maximum point of $u^\varepsilon - \varphi$ in $\mathbb{R}^N \times]0,\infty[$, then

$$\frac{\partial u^\varepsilon}{\partial t}(x_\varepsilon,t_\varepsilon) = \frac{\partial \varphi}{\partial t}(x_\varepsilon,t_\varepsilon)$$

$$Du^\varepsilon(x_\varepsilon,t_\varepsilon) = D\varphi(x_\varepsilon,t_\varepsilon)$$

$$\Delta u^\varepsilon(x_\varepsilon,t_\varepsilon) \leqslant \Delta\varphi(x_\varepsilon,t_\varepsilon)$$

In (23), this gives

$$\frac{\partial \varphi}{\partial t}(x_\varepsilon,t_\varepsilon) - \varepsilon\cdot\Delta\varphi(x_\varepsilon,t_\varepsilon) + H(x_\varepsilon,t_\varepsilon,u^\varepsilon(x_\varepsilon,t_\varepsilon),D\varphi(x_\varepsilon,t_\varepsilon)) \leqslant 0 \tag{24}$$

And we can do the same by replacing "local maximum" by "local minimum", then in (24) $\leqslant 0$ is replaced by $\geqslant 0$. So we have replace equality (23) by two families of inequalities (24), $\varphi \in C^\infty(\mathbb{R}^N \times (0,\infty))$.

Now assume that u^ε converges uniformly on compact subsets to u , $u \in C(\mathbb{R}^N \times (0,\infty))$. If we take φ in $C^\infty(\mathbb{R}^N \times (0,\infty))$ and if we assume that $u-\varphi$ has a strict local maximum point (x_o,t_o) in $\mathbb{R}^N \times]0,\infty[$, then there exist $(x_\varepsilon,t_\varepsilon)$ converging to (x_o,t_o) such that $(x_\varepsilon,t_\varepsilon)$ is a local maximum point of $u^\varepsilon-\varphi$. So we have (24) and it is easy to pass to the limit in (24). This leads to the following definition.

<u>Definition III.1.</u> Let $u \in C(\mathbb{R}^N \times [0,\infty])$.

(i) u is a viscosity subsolution of (22) if and only if

$$(25)\quad \begin{cases} \text{for all } \phi \in C^\infty(\mathbb{R}^N \times ((0,\infty)) \text{, at each local maximum point} \\ (x_o,t_o) \text{ of } u-\phi \text{ in } \mathbb{R}^N \times]0,\infty[\text{ , we have} \\ \frac{\partial\phi}{\partial t}(x_o,t_o) + H(x_o,t_o,u(x_o,t_o),D\phi(x_o,t_o)) \leq 0 \end{cases}$$

(ii) u is a viscosity supersolution of (22) if and only if

$$(26)\quad \begin{cases} \text{for all } \phi \in C^\infty(\mathbb{R}^N \times (0,\infty)) \text{, at each local minimum point} \\ (x_o,t_o) \text{ of } u-\phi \text{ in } \mathbb{R}^N \times]0,\infty[\text{ , we have} \\ \frac{\partial\phi}{\partial t}(x_o,t_o) + H(x_o,t_o,u(x_o,t_o),D\phi(x_o,t_o)) \geq 0 \end{cases}$$

(iii) u is a viscosity solution of (22) if and only if u satisfies both (25) and (26).

Remark III.1 : This definition requires only u to be continuous.

The first property we want to mention is consistency : If u is a viscosity solution of (22) and if u is differentiable at (x_o,t_o) in $\mathbb{R}^N \times]0,\infty[$, then

$$\frac{\partial u}{\partial t}(x_o,t_o) + H(x_o,t_o,u(x_o,t_o),Du(x_o,t_o)) = 0$$

so that any viscosity solution of (22) which is in $W^{1,\infty}_{loc}(\mathbb{R}^N \times (0,\infty))$ is a generalized solution of (22).

The most important property concerns uniqueness results. We need the following assumptions

$$(27)\qquad H \text{ is uniformly continuous on } \mathbb{R}^N \times (0,T) \times (-R,R) \times \overline{B}_R$$
$$(\forall R < \infty \,,\ \forall T < \infty)$$

$$(28) \qquad \begin{cases} \forall\, R < \infty\ , \quad \gamma_R \in \mathbb{R} \quad \text{such that for} \quad x \in \mathbb{R}^N\ ,\ t \in (0,\infty)\ , \\ -R \leqslant s \leqslant r \leqslant R \quad \text{we have :} \quad H(x,t,r,p) - H(x,t,s,p) \geqslant \gamma_R (r-s) \end{cases}$$

$$(29) \qquad \limsup_{\varepsilon \downarrow 0} \Big\{ |H(x,t,r,p)-H(y,t,r,p)|\ ,\ |x-y|\,(1+|p|) \leqslant \varepsilon\ ,\ |r| \leqslant R\ ,\ t \in (0,\infty) \Big\} = 0 \qquad (\forall\, R < \infty)$$

We denote by $BUC(\mathbb{R}^N \times (0,\infty))$ the space of bounded uniformly continuous functions in $\mathbb{R}^N \times (0,\infty)$. The result is the following

Theorem III.1 : Let u and v be respectively viscosity sub- and super-solution of (22) with respective initial conditions u_o and v_o.

(i) Under assumptions (27), (28), (29) and if u and v are in $BUC(\mathbb{R}^N \times (0,\infty))$, we have

$$(\star) \qquad \|(u-v)^+(\cdot,T)\|_{L^\infty(\mathbb{R}^N)} \leqslant e^{-\gamma T} \|(u_o - v_o)^+\|_{L^\infty(\mathbb{R}^N)}$$

where $\gamma = \gamma_{R_o}$, $R_o = \max\left(\|u\|_{L^\infty(\mathbb{R}^N)}\ ,\ \|v\|_{L^\infty(\mathbb{R}^N)} \right)$.

(ii) Under assumptions (27), (28) and if u and v are in $W^{1,\infty}(\mathbb{R}^N \times (0,\infty))$ then $(\star)$ holds.

This result is actually more than an uniqueness result, it is a comparaison result of Maximum principle type. Then one can obtain existence results (several of them using in an essential way the preceding comparison result) ; we do not give any detail about existence results, we refer to [19], [20], [25], [13], [2], [3].

We now explain some relations between viscosity solutions of first-order Hamilton Jacobi Equations and Geometrical Optics. We consider a particular case of (22) :

$$
(30) \qquad \begin{cases} \dfrac{\partial u}{\partial t} + H(Du) = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = u_o(x) & \text{in } \mathbb{R}^N \end{cases}
$$

One particular case of (30) is the case when $H(p) = +/-|p|$ which gives the Eikonal equation (cf. (10)). Then, it is natural (cf. Remark II.1 and [16]) to introduce the Hamiltonian system

$$
\begin{cases} \dfrac{dx(t)}{dt} = \dfrac{\partial H}{\partial p}(p(t),x(t)) \quad , & x(0) = x_o \\ \dfrac{dp(t)}{dt} = -\dfrac{\partial H}{\partial x}(p(t),x(t)) \quad , & p(0) = Du_o(x_o) \end{cases}
$$

Since H does not depend on x, we deduce

$$
p(t) = Du_o(x_o)
$$

$$
x(t) = x_o + tH'(Du_o(x_o))
$$

As long as u is smooth we have

$$
\frac{d}{dt}(u(x(t),t) = \frac{\partial u}{\partial t}(x(t),t) + \frac{dx(t)}{dt} \cdot \nabla u(x(t),t)
$$

But $p(t) = \nabla u(x(t),t)$, then

$$
\frac{d}{dt}(u(x(t),t) = -H(Du_o(x_o)) + H'(Du_o(x_o)) \cdot Du_o(x_o)
$$

Finally :

$$
(31) \qquad \begin{cases} x(t) = x_o + tH'(Du_o(x_o)) \\ u(x(t),t) = u_o(x_o) + t\,[H'(Du_o(x_o)) \cdot Du_o(x_o) - H(Du_o(x_o))] \end{cases}
$$

This method is called the <u>Characteristics method</u> for first-order Hamilton Jacobi Equations ; in fact, the Characteristics method is applied to the <u>hyperbolic system</u>

$$\frac{dp}{dt} + (H(p))_x = 0$$

where p represents $Du(x,t)$.

But as in geometrical optics, caustics form and we must be able to define u after.

From now on, we assume that <u>H is convex</u>. We consider the dual convex function of H as in (19)

$$\text{(32)} \qquad L(\dot{x}) = \sup_p ((\dot{x}|p) - H(p)) = H^\star(\dot{x})$$

and we define u by

$$\text{(33)} \qquad u(x,t) = \inf_{\xi(t)=x} \left\{ u_o(\xi(0)) + \int_o^t H^\star(\dot{\xi}(s))ds \right\}$$

But $H^\star$ is convex, so by Jensen's inequality

$$\int_o^t H^\star(\dot{\xi}(s))ds \geqslant t\, H^\star\left(\frac{1}{t} \int_o^t \dot{\xi}(s)ds\right)$$

and

$$H^\star\left(\frac{1}{t} \int_o^t \dot{\xi}(s)ds\right) = H^\star\left(\frac{\xi(t)-\xi(0)}{t}\right)$$

Then (33) becomes, setting $y = \xi(0)$

$$\text{(34)} \qquad u(x,t) = \inf_{y \in \mathbb{R}^N} \left(u_o(y) + t\, H^\star\left(\frac{x-y}{t}\right)\right)$$

which is the well-known <u>Oleinik-Lax formula</u> which gives the unique viscosity solution of (30).

<u>Remark III.1</u> : Denote by $S_H(t)u_o$ the unique viscosity solution of (30). S_H is the semi-group associated to H (cf. [7], [8], [19]). If H depends only on $|p|$, we have

$$S_{-H}(t)u_o = - S_H(t)(-u_o)$$

So that in this case the Oleinik Lax formula can be applied to concave Hamiltonian.

Using the remark above in the case of flame front propagation where $H(p) = -c|p|$, we get

$$S_H(t)\cdot u_o^{(x)} = \sup_{|y-x| \leqslant ct} (u_o(y)) \tag{35}$$

then taking $u_o(x) = \varphi(x,0)$ defined by (7), we obtain the following result.

<u>Theorem III.2</u> : Let φ defined by (6) and (7), then φ is the unique viscosity solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} - c\cdot|Du| = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \qquad (N = 2 \text{ or } 3) \\ u(x,0) = \varphi(x,0) & \text{in } \mathbb{R}^N \end{cases}$$

Let us finally remark that (35) allows any continuous function u_o ; so if $\varphi(x,0)$ is defined by (7), we can take any initial burnt region $\bar{\Omega}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$.

IV. Another approach of the problem : deterministic optimal control.

Another possibility to prove that φ is the unique viscosity solution of (36) is to use deterministic optimal control method and in particular the Dynamic Programming Principle. In fact as regard of (33), (35) or (6), the analogy with deterministic optimal control is obvious. We now present the classical form of the Dynamic Programming Principle in deterministic optimal control problems for a sake of completeness and of clarity (see [10] and [19] for more details).

We consider a system which state is described by the solution $y_x(t)$ of the following problem

$$\begin{cases} \dfrac{dy_x(t)}{dt} = b(y_x(t),v(t)) \\ y_x(0) = x \in \mathbb{R}^N \end{cases}$$

where $v \in L^\infty(0,\infty,V)$, V is separable metric space - <u>the space of Controls</u> - and b satisfies

$$\text{(37)} \qquad \begin{cases} b \in C(\mathbb{R}^N \times V) \\ \forall\, v \in V , \quad b(\cdot,v) \in W^{1,\infty}(\mathbb{R}^N) \quad \text{and} \quad \sup\limits_{v \in V} \|b(\cdot,v)\|_{1,\infty} < +\infty \end{cases}$$

Notice that (37) ensures existence and uniqueness for y_x .

Then we define the cost function $J(x,t,v(\cdot))$ by

$$J(x,t,v(\cdot)) = \int_0^t f(y_x(s),v(s))e^{-\lambda s}ds + u_0(y_x(t))e^{-\lambda t}$$

where f satisfies (37), $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ and $\lambda > 0$.

Now the optimal cost function u is defined by

$$u(x,t) = \operatorname*{Inf}_{v(\cdot)} (J(x,t,v(\cdot)))$$

The Dynamic Programming Principle is the following result :

Theorem IV.1 : We have

$$u(x,t) = \operatorname*{Inf}_{v(\cdot)} \left\{ \int_0^h f(y_x(s),v(s))e^{-\lambda s}ds + u(y_x(h),t-h)e^{-\lambda h} \right\}$$

for all $h \in (0,t)$.

Remark IV.1 : The Dynamic Programming Principle is an essential tool to prove that u is viscosity solution of

$$\frac{\partial u}{\partial t} + H(x,u,Du) = 0 \qquad \text{in } \mathbb{R}^N \times (0,\infty)$$

where

$$H(x,r,p) = \sup_{v \in V} \{b(x,v)\cdot p + \lambda\cdot r - f(x,v)\}$$

(cf. [19]).

In our context of flame propagation, it takes the following form

Theorem IV.2 : Let φ defined by (6), then we have

(38) $$\varphi(x,t) = \sup_{|y-x| \leqslant c\cdot h} \{\varphi(y,t-h)\}$$

Remark IV.2 : Let us first notice that (38) is a particular case of theorem IV.1. Let us take

$$V = \{v \in \mathbb{R}^N \ / \ |v| \leqslant 1\}$$

$$b(x,v) = v$$

$$f \equiv 0$$

and

$$\lambda = 0$$

Then to have "sup" instead of "inf", it suffices to change u_o in $-u_o$ and u in $-u$.

Remark IV.3 : Physically, (38) is obvious. It means that to have the shape of the flame front at time t , it suffices to know its shape at any time $t' < t$ and to use the laws of propagation described in part I.

Now let us show to prove that φ is a viscosity solution of (1) by using (38). We only prove that φ is a viscosity subsolution of (1), the proof that φ is a viscosity supersolution of (1) being exactly the same.

Let $\psi \in C^\infty(\mathbb{R}^N \times (0,\infty))$ and assume that $\varphi-\psi$ has a local maximum point (x_o,t_o) in $\mathbb{R}^N \times]0,\infty[$. Then

$$\varphi(x_o,t_o) - \psi(x_o,t_o) \geqslant \varphi(x,t) - \psi(x,t)$$

for $|x-x_o| + |t-t_o|$ small enough. So using (38), we have for h small enough

$$\begin{aligned} \varphi(x_o,t_o) &= \sup_{|y-x_o| \leqslant ch} (\varphi(y,t_o-h)) \\ &\leqslant \sup_{|y-x_o| \leqslant ch} (\psi(y,t_o-h) + \varphi(x_o,t_o) - \psi(x_o,t_o)) \end{aligned}$$

Hence :

$$\sup_{|y-x_o| \leqslant ch} (\psi(y,t_o-h) - \psi(x_o,t_o)) \geqslant 0$$

But since ψ is C^1

$$\psi(y,t_o-h) - \psi(x_o,t_o) = \nabla\psi(x_o,t_o)\cdot(y-x_o) - \frac{\partial\psi}{\partial t}(x_o,t_o)\cdot h + \mathcal{O}(h)$$

We deduce

$$\sup_{|y-x_o| \leqslant ch} (\nabla\psi(x_o,t_o)(y-x_o)) - \frac{\partial\psi}{\partial t}(x_o,t_o)h + O(h) \geqslant 0$$

Notice that

$$\sup_{|y-x_o| \leqslant ch} (\nabla\psi(x_o,t_o)(y-x_o)) = c\cdot h|\nabla\psi(x_o,t_o)|$$

Dividing by h and letting $h \to 0$ we obtain

$$\frac{\partial\psi}{\partial t}(x_o,t_o) - c|\nabla\psi(x_o,t_o)| \leqslant 0$$

which we wanted to prove.

V. Evolution of the flame front and viscosity solutions.

In this section we are interested in the propagation of a flame front in $\mathbb{R}^N$ (N = 2 or 3) which is given a time $t = 0$ by

$$\Gamma = \{(x,y) \in \mathbb{R}^{N-1} \times \mathbb{R} \; / \; y = u_o(x)\}$$

where u_o is at least continuous. We assume that the region above C is filled with a premixed combustible fluid and that the particles below C are burnt. Our aim is to describe the propagation of C after the ignition of the particles along (or on) C at time $t = 0$. The law of propagation is the same as the one described in part I. Our interest for this particular case is motivated by the following remark :

Remark V.1 : Let us recall the indicator function of the burnt region ; in our context, it takes the particular form

$$(39) \qquad \varphi(x,y,t) = \begin{cases} 1 & \text{if} \quad y \leqslant u_o(x) \\ 1 & \text{if} \quad (ct)^2 \geqslant \min\limits_{s \in \mathbb{R}^{N-1}} (|x-s|^2 + |y-u_o(s)|^2) \\ 0 & \text{otherwise} \end{cases}$$

This implies that the burnt region at time t is of the form

$$\Omega_t = \{(x,y) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid y \leqslant u(x,t)\}$$

And so the flame front at time t is parametrized by

$$y = u(x,t)$$

Therefore instead of taking $\varphi(x,y,t)$ as unknown function, we now take u.

As in [24], we first define the ignition curves. In this context they take the form :

$$(40)\qquad \begin{cases} X(s,t) = -c\cdot t\cdot \dfrac{u_o'(s)}{(1+(u_o'(s))^2)^{1/2}} + s \\[2ex] Y(s,t) = \dfrac{ct}{(1+(u_o'(s))^2)^{1/2}} + u_o(s) \end{cases}$$

The same problem holds : if t is small enough and if u_o is smooth we can define u by

$$(41)\qquad Y(s,t) = u(X(s,t),t)$$

but it may happen that the function $s \to X(s,t)$ fails to be a diffeomorphism (for example, fails to be injective) so that u is no more defined by (41). And this situation is closely connected to the loss of regularity of u.

Then the description of the flame front propagation is given by the following result :

Theorem V.1 : For all $t > 0$, the position of the flame front is parametrized by $y = u(x,t)$ where u is the unique viscosity solution of

$$(42)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - c\sqrt{1+|\nabla u|^2} = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = u_o(x) \end{cases}$$

for all $u_o \in C(\mathbb{R}^N)$.

Proof of theorem V.1 : The proof is particularly easy using the "weak" formulation given by (39). In fact :

$$u(x,t) = \max\left\{y \in \mathbb{R} \;\middle|\; \min_{s\in\mathbb{R}^{N-1}} (|x-s|^2+|y-u_o(s)|^2) \leq (ct)^2\right\}$$

Then we deduce that :

$$(43)\qquad u(x,t) = \sup_{|y-x|\leq ct}\left\{u_o(y) + \sqrt{(ct)^2 - |y-x|^2}\right\}$$

We leave it to the reader to check that this is exactly the Oleinik-Lax formula which gives the unique viscosity solution of (42) in the setting of remark III.1. This ends the proof of theorem V.1.

Remark V.2 : Let us remark that ignition curves given by (40) and as regard of (31) represent exactly the construction of characteristics for (43) used in the study of first order Hamilton Jacobi equations. For introduction and details on the characteristics method we refer, for example, to R. Courant and D. Hilbert [6], F. John [14,15], P.D. Lax [18], P.L. Lions [19] and H. Rund [23].

We now consider the following problem : when u_o is not differentiable at one point, how can we describe the propagation ?

To answer this question, we consider u_o concave. If u_o is smooth the characteristics method defines u at each point but if u_o is not differentiable at one point (take $u_o(x) = -|x|$) the mapping $s \to X(s,t)$, which is defined only at the point of differentiability of u_o , fails to be surjective. Intuitively, there is not enough ignition curves. We are going to define "generalized ignition curves" to fill in the empty regions. For that, we need the following definition :

Definition V.1 : Let $v \in C(\mathbb{R}^N)$, the superdifferential of v at $x_o \in \mathbb{R}^N$ denoted by $D^+v(x_o)$ is the set (possibly empty) defined by

(44) $$D^+v(x_o) = \left\{p \in \mathbb{R}^N \mid \limsup_{x \to x_o} \frac{v(x)-v(x_o)-(p|x-x_o)}{|x-x_o|} \leq 0\right\}$$

Now we can define "generalized ignition curves"

(45) $$(X,Y)(s,t) = \left\{\left(s - \frac{cpt}{\sqrt{1+|p|^2}}\ , u_o(s) + \frac{c\cdot t}{\sqrt{1+|p|^2}}\right)\ ,\ p \in D^+u_o(s)\right\}$$

Each point gives a set of ignition curves. Let us just mention that if u_o is differentiable at x_o, $D^+u_o(x_o) = \{Du_o(x_o)\}$. Our result is the following :

Theorem V.2 : Let $u_o \in W^{1,\infty}(\mathbb{R}^{N-1})$ concave and let $t > 0$. For all $x \in \mathbb{R}^{N-1}$, there exists one and only one couple $(s,p) \in (\mathbb{R}^{N-1})^2$ such that

(45) $$x = s - \frac{cpt}{\sqrt{1+|p|^2}} \quad \text{and} \quad p \in D^+u_o(s)$$

and then

(46) $$u(x,t) = u_o(s) + \frac{ct}{\sqrt{1+|p|^2}}$$

Remark V.3 : This result shows that the directions of propagation are those which are normal to one element of the superdifferential.

Remark V.4 : For $u_o \in W^{1,\infty}(\mathbb{R}^{N-1})$, the existence of (s,p) still holds but, in general, there is no uniqueness (take $u_o(x) = |x|$). But we then have :

(47) $$u(x,t) = \sup\left\{u_o(s) + \frac{ct}{\sqrt{1+|p|^2}} \;/\; x = s - \frac{cpt}{\sqrt{1+|p|^2}},\ s \in \mathbb{R}^{N-1},\ p \in D^+u_o(s)\right\}$$

The proof of this claim is essentially based on the ideas of the proof of theorem V.2.

Remark V.5 : This construction of "generalized ignition curves" is similar to the construction given by J. Sethian in [24] to study the evolution of a convex, non-smooth flame front - i.e. by reparametrization of the initial curve -. Remind that a convex front is a front for which the burnt region is convex ; this corresponds to u_o concave.

Remark V.6 : In our case, we can answer a question formulated by J. Sethian in [24]. The problem is the following : by J. Sethian method or by the method of theorem V.2, we are able to describe the evolution of a convex, non-smooth flame front. Now if we take any sequence u_o^ε which converges uniformly to u_o , does the sequence u^ε of associated propagating flame front converges, at least pointwise, to u ? Of course, the answer is easy by looking at (43). But we can also argue by using the comparaison result for viscosity solutions (see section III) which gives

$$\|u^\varepsilon - u\|_{L^\infty(\mathbb{R}^N \times (0,\infty))} \leqslant \|u_o^\varepsilon - u_o\|_{L^\infty(\mathbb{R}^N)}$$

Physically, the inequality above may be interpreted as a stability property of propagating flame fronts. (See section VI, for more details).

Proof of Theorem V.2 : We know that u is given by (43)

$$u(x,t) = \sup_{|y-x| \leqslant ct} \left\{ u_o(y) + \sqrt{(ct)^2 - |y-x|^2} \right\}$$

First we claim that the supremum is not achieved on the circle $\{y \mid |y-x| = ct\}$. Indeed, assume on the contrary that the supremum is achieved at y_o satisfying $|y_o - x| = ct$, then

$$u_o(y_o) \geqslant u_o(y) + \sqrt{(ct)^2 - |y-x|^2}$$

Since $u_o \in W^{1,\infty}(\mathbb{R}^{N-1})$, we get

$$(ct)^2 - |y-x|^2 \leqslant \|Du_o\|_\infty^2 \cdot |y-y_o|^2$$

Take y such that $y-x = \alpha(y_o - x)$ with $\alpha \in (0,1)$ then we have $|y-y_o| = (1-\alpha)|y_o - x|$ and the inequality above gives

$$(1-\alpha^2)|y_o - x|^2 \leqslant \|Du_o\|^2 \cdot (1-\alpha)^2 |y_o - x|^2$$

Dividing by $(1-\alpha)^2$ and letting $\alpha \to 1$, we get a contradiction. Moreover $y \to ((ct)^2-|y-x|^2)^{1/2}$ is strictly concave on $\{y \mid |y-x| \leqslant ct\}$, hence the supremum is achieved at only one point s

$$(u_o(s) + ((ct)^2-|s-x|^2)^{1/2} \geqslant u_o(y) + ((ct)^2-|y-x|^2)^{1/2}$$

Therefore

$$u_o(y) - u_o(s) + ((ct)^2-|y-x|^2)^{1/2} - ((ct)^2-|s-x|^2)^{1/2} \leqslant 0$$

letting $y \to s$, we obtain that p given by

$$p = (s-x) \; ((ct)^2-|s-x|^2)^{1/2}$$

belongs to $D^+u_o(s)$. Then a simple computation gives

$$|s-x|^2 = \frac{|p|^2 \cdot (ct)^2}{1 + |p|^2}$$

and

$$((ct)^2-|s-x|^2)^{1/2} = \frac{ct}{\sqrt{1+|p|^2}}$$

Finally

$$x = s - \frac{cpt}{\sqrt{1+|p|^2}} \qquad \text{and} \qquad u(x,t) = u_o(s) + \frac{ct}{\sqrt{1+|p|^2}}$$

This proves the existence of (s,p) .

Uniqueness is a consequence of the following lemma :

<u>Lemma V.1</u> : Let $u_o \in W^{1,\infty}(\mathbb{R}^{N-1})$ concave. The superdifferential of u_o is decreasing in the sense that

$$\forall\, s_1,s_2 \in \mathbb{R}^{N-1} \;,\; \forall\, p_1 \in D^+u_o(s_1) \;,\; \forall\, p_2 \in D^+u_o(s_2) \;, \quad \text{we have}$$

$$(p_1-p_2|s_1-s_2) \leqslant 0$$

We end the proof of theorem V.2 using lemma V.1. Assume that there exist s_1, s_2, p_1 and p_2 such that $p_1 \in D^+u_o(s_1)$, $p_2 \in D^+u_o(s_2)$ which satisfy

$$x = s_1 - \frac{ct \cdot p_1}{\sqrt{1+|p_1|^2}} = s_2 - \frac{ct \cdot p_2}{\sqrt{1+|p_2|^2}}$$

with $s_1 \neq s_2$.

Then

$$s_1 - s_2 = ct \cdot \left| \frac{p_1}{\sqrt{1+|p_1|^2}} - \frac{p_2}{\sqrt{1+|p_2|^2}} \right|$$

Using the convex function $f = p \to \sqrt{1+|p|^2}$, we may write the equality above as

$$s_1 - s_2 = ct \ [f'(p_1) - f'(p_2)]$$

And so

$$(s_1 - s_2 | p_1 - p_2) = ct \ \left(f'(p_1) - f'(p_2) | p_1 - p_2\right)$$

Moreover f satisfies

$$\left(f'(p_1) - f'(p_2) | p_1 - p_2\right) \geqslant [\max \ (f(p_1), f(p_2))]^{-3} \cdot |p_1 - p_2|^2$$

This inequality contradicts lemma V.1 since $p_1 \neq p_2$ because $s_1 \neq s_2$.
We omit the proof of lemma V.1 which is a classical and easy result of convex analysis.

<u>Remark V.7</u> : A similar result holds for general convex initial burnt regions. Indeed, let Ω a convex unitial burnt region and let $\Gamma = \partial\Omega$. For each $x_o \in \Gamma$, we can define a set of "generalized normals" by

$$N(x_o) = \left\{ q \in \mathbb{R}^N \mid \overline{\lim_{\substack{y \to x_o \\ y \in \Gamma}}} \left(\frac{y - x_o}{|y - x_o|} \middle| q \right) \leqslant 0 \right\}$$

Now recall that the propagating flame front at time t is given by

$$\Gamma_t = \{x \in \mathbb{R}^N \ / \ d(x,\overline{\Omega}) = ct\}$$

Then one shows easily that for each point $x \in \Gamma_t$, there exists one and only one couple (x_o,q_o) such that $x_o \in \Gamma$, $q_o \in N(x_o)$, $|q_o| = 1$ and

$$x = x_o + ct \cdot q_o$$

This means that the directions of propagation are those of the "generalized normals".

VI. Results based on the viscosity solutions formulation.

In this section, we make some remarks using the new formulation of the evolution of the flame front by viscosity solutions.

1. Stability of the flame front and comparison results.

We consider the formulation of part V, where the unknown function is directly the propagating flame front. If we take two initial burnt region $\Omega_1 = \{y \leq u_o^1(x)\}$ and $\Omega_2 = \{y \leq u_o^2(x)\}$, at each time $t > 0$ the associated propagating flame fronts are parametrized by $y = u^1(x,t)$ and $y = u^2(x,t)$ where u^1 and u^2 are the viscosity solutions of (42) respectively associated to u_o^1 and u_o^2 . Then by comparison results, (see section III) we have

$$\|u^2-u^1\|_{L^\infty(\mathbb{R}^N \times (0,\infty))} \leq \|u_o^2-u_o^1\|_{L^\infty(\mathbb{R}^N)} \qquad (N = 1 \text{ or } 2)$$

This inequality may be interpreted as a "stability" property of the propagating flame fronts : when two flame fronts are close to each other at time $t = 0$, they remain close.

This property is not correct for general initial burnt regions. For instance, take $\Omega_1 = \{(x,y) \in \mathbb{R}^2 \mid a \leq x^2+y^2 \leq 1\}$, $(0 < a < 1)$ and take $\Omega_2 = \{(x,y) \in \mathbb{R}^2 \mid a+\varepsilon \leq x^2+y^2 \leq 1\}$ (ε small). At time $t = 0$, $\partial\Omega_1$ and $\partial\Omega_2$ are close together in the sense that each point of $\partial\Omega_1$ is within a distance of ε to a point $\partial\Omega_1$ and conversely. But at time $t = a/c$, we have $\Omega_1^t = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1+ct\}$ and $\Omega_2^t = \{(x,y) \in \mathbb{R}^2 \mid \varepsilon \leq x^2+y^2 \leq 1+ct\}$. So, $\partial\Omega_2^t$ is not close to $\partial\Omega_1^t$. Nevertheless, one may consider that this approach is not satisfactory since in our example Ω_1^t and Ω_2^t remain close.

If we define the distance between Ω_1 and Ω_2 by

$$d(\Omega_1,\Omega_2) = \max\left\{\sup_{x\in\Omega_1}(d(x,\Omega_2))\ ;\ \sup_{x\in\Omega_2}(d(x,\Omega_1))\right\}$$

the correct stability property is given by the following result.

Theorem VI.1 : Let Ω_1 and Ω_2 be two closed initial burnt regions and let Ω_1^t and Ω_2^t be the associated burnt regions at time $t > 0$. We assume that $d(\Omega_1,\Omega_2) \leqslant \varepsilon < 1$, then

$$d(\Omega_1^t,\Omega_2^t) \leqslant d(\Omega_1,\Omega_2)$$

Proof of theorem VI.1 : We need the following lemma.

Lemma VI.1 : Let Ω be a closed initial burnt region and let Ω^t be the associated burnt region at time $t > 0$. Let φ be the mass fraction of burnt gas associated to Ω by (6) and (7). We have :

$$\text{if } d(x,\Omega) \geqslant ct\ ,\quad \varphi(x,t) = (1-d(x,\Omega^t))^+ = (1-d(x,\Omega)+ct)^+$$

First we prove Theorem VI.1 by using Lemma VI.1. Let φ_1 and φ_2 associated to Ω by (6) and (7). Since φ_1 and φ_2 are viscosity solutions of (1), we obtain, by comparison results for viscosity solutions :

$$\|\varphi_1-\varphi_2\|_{L^\infty(\mathbb{R}^N\times(0,\infty))} \leqslant \|\varphi_1(\cdot,0)-\varphi_2(\cdot,0)\|_{L^\infty(\mathbb{R}^N)}$$

We leave it to the reader to prove that $d(\Omega_1,\Omega_2) = d(\Omega_1^c,\Omega_2^c)$.

Using this property and (7), we obtain easily that

$$\|\varphi_1(\cdot,0)-\varphi_2(\cdot,0)\|_{L^\infty(\mathbb{R}^N)} \leqslant d(\Omega_1,\Omega_2)$$

Then for all $t > 0$ and all $x \in \mathbb{R}^N$, we have

$$|\varphi_1(x,t)-\varphi_2(x,t)| \leq d(\Omega_1,\Omega_2) \leq \varepsilon < 1$$

Therefore, if $x \in \Omega_1^t$, we can estimate $\varphi_2(x,t)$ by

$$\varphi_2(x,t) \geq 1 - d(\Omega_1,\Omega_2)$$

Using Lemma VI.1, we deduce that either $x \in \Omega_2^t$ or

$$d(x,\Omega_2^t) \leq d(\Omega_1,\Omega_2)$$

Finally

$$\sup_{x \in \Omega_1^t} (d(x,\Omega_2^t)) \leq d(\Omega_1,\Omega_2)$$

In the same way, we obtain

$$\sup_{x \in \Omega_2^t} (d(x,\Omega_1^t)) \leq d(\Omega_1,\Omega_2)$$

So

$$d(\Omega_1^t,\Omega_2^t) \leq d(\Omega_1,\Omega_2)$$

which we wanted to prove.

Now we give the proof of lemma VI.1.

Proof of Lemma VI.1 : φ is given by (6) and (7) then

$$\varphi(x,t) = \sup_{|y-x| \leq ct} ((1-d(y,\Omega))^+ + d(y,\Omega^c))$$

Since we have $d(x,\Omega) \geq ct$, all the points y which satisfy $|y-x| \leq ct$ are in $\overline{\Omega}^c$. Then a simple computation yields

$$\varphi(x,t) = \left[1 - \inf_{|y-x| \leq ct} (d(y,\Omega))\right]^+$$

This equality obviously gives

$$\varphi(x,t) = (1 - d(x,\Omega) + ct)^+$$

Since $\Omega_t = \{y \in \mathbb{R}^N \ / \ d(y,\Omega) \leqslant ct\}$, it is easy to see that

$$d(x,\Omega) = ct + d(x,\Omega_t)$$

And the proof is complete.

2. Asymptotic behavior when $t \to +\infty$:

In the context of part III, where the unknown function is the mass fraction of burnt gas, the formula

$$\varphi(x,t) = \sup_{|y-x| \leqslant ct} (\varphi(x,0)) ,$$

where $\varphi(x,0)$ is given by (7), shows that the flame front at time $t > 0$ is given by $\{x \in \mathbb{R}^N \ / \ d(x,\Omega) = ct\}$; then for such x we have, if Ω is bounded

$$\frac{|x-x_o|-|x_o|}{ct} \leqslant \frac{|x|}{ct} \leqslant \frac{|x-x_o|+|x_o|}{c\ t} \qquad \text{for all } x_o \in \Omega$$

Taking the infimum, estimating $|x_o|$ by $B = \sup_{\Omega} |x_o|$, we obtain

$$1 - \frac{B}{ct} \leqslant \frac{|x|}{ct} \leqslant 1 + \frac{B}{ct}$$

We conclude as in [24] that the shape of the flame front becomes circular.

In the context of part V, using results proved by P.L. Lions [19] or G. Barles [1], it is easy to see that

$$u(x,t) - ct \to \sup_{\mathbb{R}^{N-1}} u_o \qquad \text{uniformly on compact subsets}$$

when $t \to +\infty$. So, the flame front becomes flatter. This behavior is quite different from the one described in [24] (and above). It is a consequence of the shape of the initial burnt region. Moreover let us observe, in another way that the example when $u_0(x) = \sqrt{1-|x|^2}$ if $|x| \leq 1$ and 0 otherwise, can not be viewed as the restriction to the half-space $\{y \geq 0\}$ of the propagation of a flame front whose initial condition would be the unit circle !

Remark VI.1 : We can take for more general laws of propagation for the flame front. Indeed let us just consider the following example : in the context of part V, we assume that the velocity c depends on x . It is easy to see that the flame front u satisfies

$$(50) \qquad \frac{\partial u}{\partial t} - c(x)\sqrt{1+|\nabla u|^2} = 0 \qquad \text{in } \mathbb{R}^N \qquad (N = 2 \text{ or } 3)$$

If $c \in W^{1,\infty}(\mathbb{R}^N)$, we have for (50) comparison results (see section III and [8], [7], [19]) and existence results (see [19], [20], [25], [13], [2], [3]) for viscosity solutions.

Remark VI.2 : Numerical Methods.

To solve (36) or (42) various numerical methods exist ; essentially

- Vanishing viscosity methods, i.e. approximation by

$$\frac{\partial u^\varepsilon}{\partial t} - \varepsilon\Delta u^\varepsilon - \begin{cases} \sqrt{1+|\nabla u^\varepsilon|^2} \\ |\nabla u^\varepsilon| \end{cases} = 0$$

- Hyperbolic schemes.

The reader can find details about these methods in [4], [9], [11], [12], [21], [22], [25].

Bibliography :

[1] Barles, G., Asymptotic behavior of viscosity solutions of first order Hamilton Jacobi Equations. (Work in preparation).

[2] Barles, G., Existence results for first order Hamilton Jacobi Equations. (To appear in Annales de l'Institut Henri Poincaré. Analyse non linéaire).

[3] Barles, G., Remarques sur des résultats d'existence pour les équations de Hamilton Jacobi du premier ordre. (Submitted to Annales de l'Institut Henri Poincaré. Analyse non linéaire).

[4] Bellman, R., Adaptive control processes. A guided tour. Princeton Univ. Press, Princeton, N.J. (1961).

[5] Bensoussan, A., Lions, J.L., and Papanicolaou, G., : Asymptotic analysis for periodic structures ; North Holland Amsterdam (1978).

[6] Courant, R. and Hilbert, D., : Methods of Mathematical Physics, Vol. I and II, Wiley, New-York (1953, 1962).

[7] Crandall, M.G., Evans, L.C. and Lions, P.L., : Some properties of viscosity solutions of Hamilton Jacobi equations, Trans. Amer. Math. Soc., to appear.

[8] Crandall, M.G., and Lions, P.L., : Viscosity solutions of Hamilton Jacobi equations, Trans. Amer. Math. Soc. 277 (1983), 1-42.

[9] Crandall, M.G. and Lions, P.L., : Two approximations of solutions of Hamilton Jacobi equations. M.R.C. report 2431, University of Wisconsin. Madison (1982).

[10] Fleming, W.H. and Rishel, R.W. : Deterministic and Stochastic optimal control. Springer, Berlin (1975).

[11] Gonzalez, R. and Rofman, E. :

[12] Gonzalez, R. and Rofman, E., : An algorithm to obtain the maximum solution of Hamilton Jacobi Equations (In Optimization Techniques. Ed. J. Stocs, Springer, Berlin, 1978).

[13] Ishii, H., Remarks on Existence of viscosity solutions of Hamilton Jacobi Equations. Bull. Facul. Sci. & Eng. CHUO Univ. Vol. 26 (1983) pp. 5-24.

[14] John, F., Partial differential Equations, Courant Institute, New-York (1953).

[15] John, F., Partial differential Equations, 2^{nd} edition, Springer, Berlin (1975).

[16] Landau, L. and Lifchitz, E. : Mecanique. Edition Mir. Moscou (1969).

[17] Landau, L. and Lifchitz, E. : Théorie des champs. Edition Mir. Moscou (1970).

[18] Lax, P.D., Partial differential Equations, Courant Institute, New-York (1951).

[19] Lions, P.L., Generalized solutions of Hamilton Jacobi equations. Pitman, London (1982).

[20] Lions, P.L., Existence results for first order Hamilton Jacobi equations. To appear in Rie. Math..

[21] Lions, P.L. and Mercier, B. : Approximation numérique des équations de Hamilton-Jacobi-Bellman. R.A.I.R.O. 14 151-166 (1980).

[22] Quadrat, J.P., Analyse numérique de l'équation de Bellman stochastique. Rapport Laboria, 140, INRIA Rocquencourt.

[23] Rund, H., The Hamilton Jacobi theory in the calculus of variations. Van Nostrand, Princeton, N.J. (1966).

[24] Sethian, J.A., An analysis of flame propagation. Ph. D Thesis. Lawrence Berkeley Laboratory, University of California.

[25] Souganidis, P.E., : Ph. D Thesis, University of Wisconsin, Madison.

G. BARLES

Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique